Lecture 22

Two Constraints

Suppose now that we want to extremize $f(x,y,z)$ subject to two constraints: $g(x,y,z)=k$ and $h(x,y,z)=c$. What we're doing here is extremizing f on the curve of intersection of the two level surfaces. Now, ∇f is still perpendicular to the curve of intersection, but ∇f might not be perpendicular to both $g=k$ & $h=c$. However, since ∇g & ∇h are perpendicular to the curve of intersection, we have that $\nabla f(P) = \lambda \nabla g(P) + \mu \nabla h(P)$ where P is an extreme point. So, in two constraints, we want to solve the system:

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) = k \\ h(x,y,z) = c. \end{cases}$$

Recall that a conic section (circle, ellipse, hyperbola, parabola) can be obtained as the intersection of a cone and a plane.

Ex: Find the points on the conic section determined by $z^2 = x^2 + y^2$ and $z = x + y + 2$ which are closest to the origin.

Sol: Let $g = x^2 + y^2 - z^2$ & $h = x + y - z + 2$

We want to minimize the function

$$f(x,y,z) = d^2 = x^2 + y^2 + z^2$$

subject to $g = 0$, $h = 0$. So, we want to solve

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = 0 \\ h = 0. \end{cases}$$

$\nabla f = \langle 2x, 2y, 2z \rangle$, $\nabla g = \langle 2x, 2y, -2z \rangle$, $\nabla h = \langle 1, 1, -1 \rangle$, so

$$\begin{cases} 2x = \lambda 2x + \mu & (1) \\ 2y = \lambda 2y + \mu & (2) \\ 2z = \lambda(-2z) - \mu & (3) \\ z^2 = x^2 + y^2 & (4) \\ z = x + y + 2 & (5) \end{cases}$$

Using (1) & (2), we can eliminate μ:

$\mu \overset{(1)}{=} 2x - 2\lambda x \overset{(2)}{=} 2y - 2\lambda y$

$\Rightarrow x(1-\lambda) = y(1-\lambda) \Leftrightarrow (x-y)(1-\lambda) = 0.$

So, $x = y$ or $\lambda = 1$.

$\underline{\lambda = 1}$: By the above $\mu = 2x - 2\lambda x = 2x - 2x = 0$.

If $\lambda = 1$ & $\mu = 0$, by (3) we have $2z = -2z \Rightarrow z = 0$. Then, by (4) $\Rightarrow x^2 + y^2 = 0 \Rightarrow x = y = 0$. But $x = y = z = 0$ contradicts (5).

So $\lambda \neq 1$. Thus $x=y$. Then we have:

$④ \Rightarrow z^2 = 2x^2$: $④'$

$⑤ \Rightarrow z = 2x+2$: $⑤'$

Plug $⑤'$ into $④'$: $(2x+2)^2 = 2x^2$

$\Rightarrow 4x^2+8x+4 = 2x^2 \Rightarrow 2x^2+8x+4 = 2(x^2+4x+2) = 0$

$\Rightarrow x^2+4x+2=0 \Rightarrow x = \dfrac{-4 \pm \sqrt{16-8}}{2} = \dfrac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$

We also have $x=y$. By $⑤'$, if

$\underline{x=-2+\sqrt{2}}$: $z = 2x+2 = -4+2\sqrt{2}+2 = -2+2\sqrt{2}$

$\underline{x=-2-\sqrt{2}}$: $z = 2x+2 = -4-2\sqrt{2}+2 = -2-2\sqrt{2}$

So, the candidate points are

$P = (-2+\sqrt{2}, -2+\sqrt{2}, -2+2\sqrt{2})$ & $(-2-\sqrt{2}, -2-\sqrt{2}, -2-2\sqrt{2}) = Q$

Point	Value
P	$24-16\sqrt{2}$ ← closest point.
Q	$24+16\sqrt{2}$

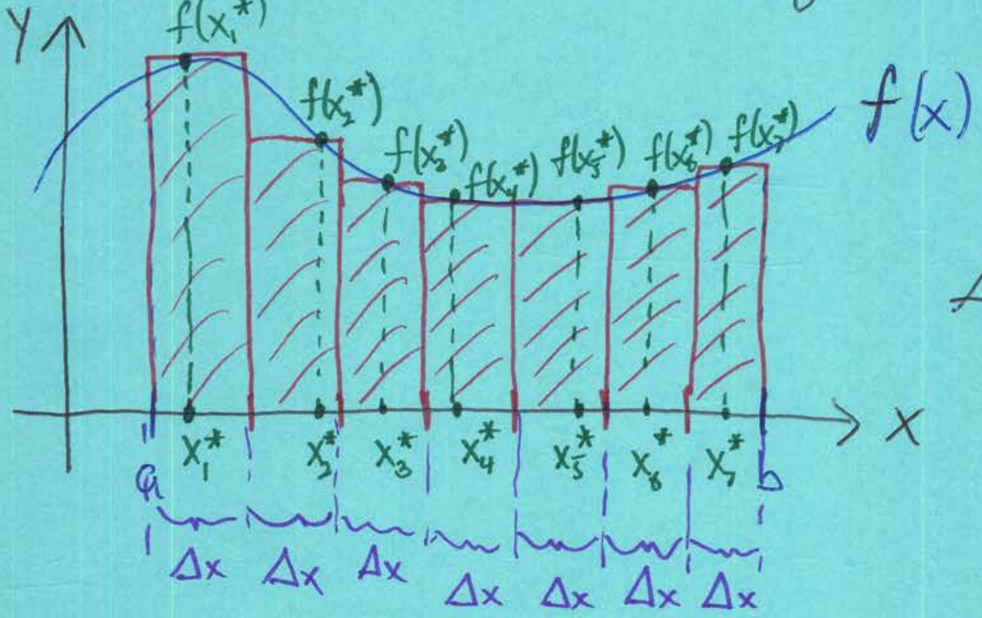
What is the other point? Since P has a positive z-value & Q has a negative z-value, the plane hits both "nappes" of the cone, hence must be a hyperboloid. Q is the other vertex.

◇

15.1 - Double Integrals Over Rectangles.

Let the Calc II analogies finally begin!

Recall the definition of an integral:



$$\Delta x = \frac{b-a}{n}$$

$$A = \int_a^b f(x)\,dx \approx \sum_{i=1}^{7} f(x_i^*)\Delta x \quad \text{(approx)} \quad \begin{pmatrix}\text{area under} \\ f(x)\end{pmatrix}$$

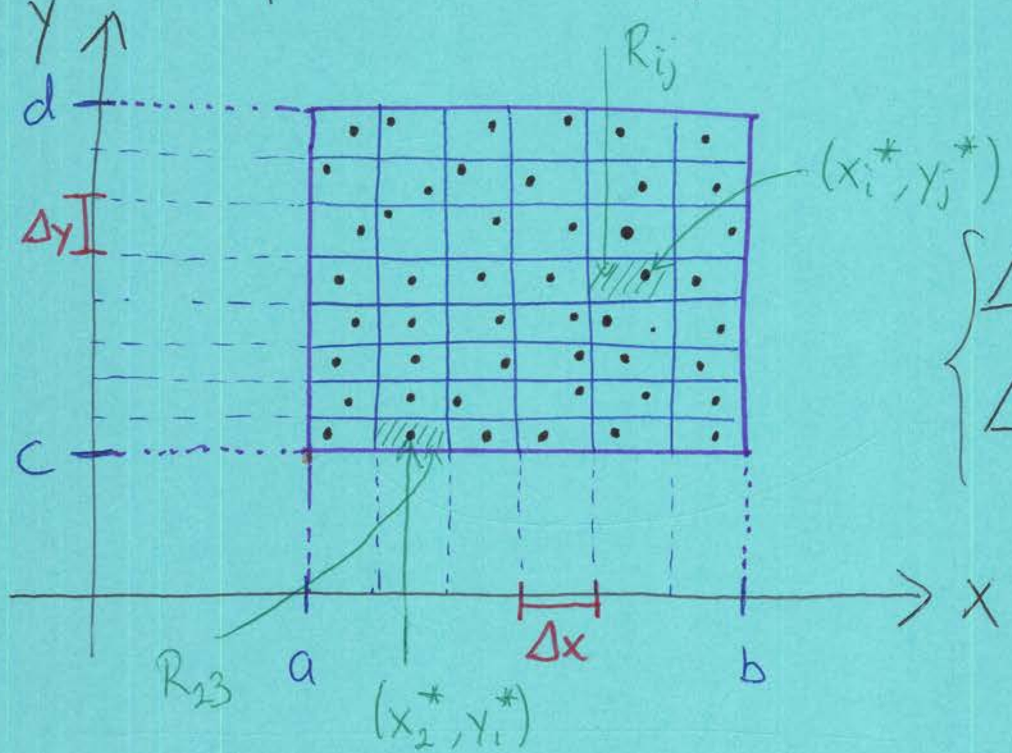
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \quad \text{(exact)}$$

Double Integrals:

We start with the simplest double integral: over a rectangle.

Let $R = [a,b] \times [c,d]$ be a rectangle, and consider a function $f(x,y)$ that contains R in its domain.

Our first step is to cut up the rectangle:



$$\begin{cases} \Delta x = \dfrac{b-a}{m} \\ \Delta y = \dfrac{d-c}{n} \end{cases}$$

Let's assume, for now, that $f(x,y) \geq 0$ on R. In each subrectangle R_{ij}, we choose a sample point (x_i^*, y_j^*), then a crude estimate of the volume bounded by the xy-plane & $f(x,y)$ over R_{ij} is

$$(\text{height})(\text{area of base}) \approx f(x_i^*, y_j^*)\Delta x \Delta y = f(x_i^*, y_j^*)\Delta A$$

$$\Delta A = \Delta x \Delta y.$$

Do this for each R_{ij}, and sum them up to approximate the volume under $f(x,y)$ over R:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A$$ (This is our <u>double Riemann sum.</u>)

Now, to get the true volume, we take the division (or partition) of the rectangle to get finer and finer: i.e.,

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A.$$

There's no reason we actually need $f(x,y) \geq 0$ in general. So, the definition we seek is:

<u>Def</u>: The <u>double integral</u> of f over the rectangle R is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A,$$

if the limit exists.

<u>Def</u>: A function $f(x,y)$ is <u>integrable</u> if for every $\varepsilon > 0$ there is a number N such that for any choice of $m, n > N$ and any choice of sample points (x_i^*, y_j^*):

$$\left| \iint_R f(x,y)\,dA - \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A \right| < \varepsilon.$$

Facts:

1) If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)\,dA$$

2) $$\iint_R [f(x,y) + g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

3) $$\iint_R c f(x,y)\,dA = c \iint_R f(x,y)\,dA \quad (c \text{ a constant})$$

4) If $f(x,y) \geq g(x,y)$ for all (x,y) in R:

then: $$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA.$$